Math 122 Friday, October 14

V a finite dimensional vector space over a field F

$T: V \to W$ $\quad \ker T \subset V$ $\quad\quad \operatorname{Im} T \subset W$

$\quad\quad \overset{\shortparallel}{\{v \in V \mid Tv = 0 \text{ in } W\}} \quad \overset{\shortparallel}{\{w \in W \mid w = T(v)\ v \in V\}}$

$\dim \ker T =$ nullity of T $\quad \dim \operatorname{Im} T =$ rank of T

Prop $\dim V = \dim \ker T + \dim \operatorname{im} T$

Pf: Let $\{e_1, \dots, e_m\}$ be a basis for $\ker T$. These are lin indep in V so can be extended to a basis $\{e_1, \dots, e_m, f_1, \dots, f_{n-m}\}$ $n = \dim V$. Guess the vectors $Tf_1, \dots, Tf_{n-m}$ form a basis for $\operatorname{im} T \subset W$. Spans: $Tv = T(\sum a_i e_i + \sum b_i f_i) = \sum a_i \underset{0}{\underset{\shortparallel}{T e_i}} + \sum b_i T f_i$. Lin indep: $0 = \sum b_i T(f_i) = T(\underset{v}{\underset{\shortparallel}{\sum b_i f_i}}) \Rightarrow v \in \ker T \Rightarrow v = \sum a_i e_i$ as e_i's give a basis of $\ker T$. Then $\sum a_i e_i - \sum b_i f_i = 0 \Rightarrow$ all coefficients are zero.

Corollary T is an isomorphism $\Leftrightarrow \ker T = 0 \Leftrightarrow \operatorname{im} T = V \Leftrightarrow \det T = \det A \neq 0$ for any choice of basis giving the matrix A.

Choose a basis $\{e_1, \dots, e_n\}$ of V. T gives an $n \times n$ matrix $A = (a_{ij})$ where the jth column is $Te_j = \sum a_{ij} e_i$

$G = GL_n(F) = \{$inv. $n \times n$ matrices A over F under mult$\} = GL(V) = \{$invertible linear maps $T \to T$ under composition$\}$

$e = Id.$

Now say $T: V \to W$ $\quad \{e_1, \dots, e_m\}$ a basis for $\ker T$ $\{f_1, \dots, f_{n-m}\}$ completes to basis of V

$\{g_1, \dots, g_k\}$ a basis for W

$g_1 = Tf_1, \dots, g_{n-m} = Tf_{n-m}, g_{n-m+1}, \dots, g_k$

Then $A = \left(\begin{array}{c|c} 1 \dots 1 & 0 \\ \hline 0 & 0 \end{array}\right) = \begin{pmatrix} I_{n-m} & 0 \\ 0 & 0 \end{pmatrix}$

Say $T: V \to V$ has matrix A wrt basis $\{e_1, \dots, e_n\}$

A^* wrt basis $\{e_1^*, \dots, e_n^*\}$

Let $P =$ matrix of transformation $V \xrightarrow{\sim} V$ $e_i \mapsto e_i^*$ wrt basis $\{e_i\}$ of V. So $e_1^* = \sum a_i e_i \Rightarrow$

$P\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Then $A^* = P^{-1} A P$

$\Rightarrow \det A^* = \det P^{-1} \det A \det P = \frac{1}{\det P} \det A \det P = \det A.$

Note $\det P \neq 0$ because P is an isomorphism of vector spaces.

We can find a 'better' invariant than $\det T$.

Note $\operatorname{Tr} A = \operatorname{Tr} A^* =: \operatorname{Tr} T$ $\Rightarrow$ put together with $\det A$ in a polynomial of degree $n = \dim V$ with coefficients in F = <u>characteristic polynomial</u> of T

Consider the linear map $(xI - T): V \to V$ $\quad v \mapsto xv - Tv \quad x \in F$
$xI - A^* = P^{-1}(xI - A)P$.

$$\det(xI - T) = \det(xI - A) = x^n - \operatorname{Tr} T\, x^{n-1} + \cdots + (-1)^n \det(T)$$

e.g. T has matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(xI - T) = \det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} = (x-a)(x-d) - bc$
$= x^2 - \underset{\operatorname{Tr}}{(a+d)}x + \underset{\det}{(ad-bc)}$

$V = F^2 \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id \qquad$ char poly is $x^2 - 2x + 1 = (x-1)^2$ for both
but $B \neq P^{-1}AP$ for any P.

$T(e_1) = e_1 \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$T(e_2) = e_1 + e_2$

How do we get <u>roots</u> of char poly $f(x)$ of T?
$c \in F$ $\; f(c) = 0 \Rightarrow \det(cI - T) = 0 \Rightarrow cI - T$ is not invertible so kernel is non-zero
$\exists\, v \neq 0$ in V s.t. $(cI - T)v = 0 = cv - Tv \Rightarrow$
$Tv = cv$, c an <u>eigenvalue</u> and v an <u>eigenvector</u>

Note for each distinct root (max number is n) you get a distinct eigenvalue + eigenvector.

$T: V \to V$ Suppose $f(x)$ = char poly $= (x - c_1)\cdots(x - c_n)$ over F, $c_1 \neq c_2 \neq \cdots \neq c_n$
factors completely with <u>distinct</u> roots.

Then I can find $\{v_1, \ldots, v_n\}$ $\; Tv_i = c_i v_i$ $\quad$ <u>Claim</u> These form a basis!

Then T wrt this basis gives a matrix $A = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$ is diagonalizable.

Note $\det xI - A = \det\begin{pmatrix} x - c_1 & & \\ & \ddots & \\ & & x - c_n \end{pmatrix}$

e.g. matrix whose char poly has no roots in $\mathbb{R}$ $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad f(x) = x^2 + 1$

Note: map is rotation by $90°$